Vibration and Damping Analysis of a Multilayered Cylindrical Shell, Part I: Theoretical Analysis

Naiyar Alam*
Aligarh Muslim University, Aligarh, India
and
N. T. Asnani†
Indian Institute of Technology, Delhi, India

The governing equations of motion for the nonaxisymmetric and axisymmetric variational of a general multilayered cylindrical shell having an arbitrary number of orthotropic material layers have been derived using variational principles. The refined analysis considers bending, extension, and shear deformations in all layers of a multilayered cylindrical shell, including rotary and longitudinal translatory as well as transverse inertias. The solution for a radially simply supported shell has been obtained and the procedure for determining the damping effectiveness in terms of the system loss factor for all families of the modes of vibration in a multilayered shell with elastic and viscoelastic layers is reported. Numerical results are reported in Part II of the paper.

Nomenclature

$E_{x,i}$ = Young's modulus of ith elastic layer of multilayered shell along axial direction; in-phase component of Young's modulus of ith viscoelastic layer of multilayered shell along axial direction $(i=1,2,3,...,n)$

$E_{\phi,i}$ = Young's modulus of ith layer of multilayered shell along circumferential direction; in-phase component of Young's modulus of ith viscoelastic layer of multilayered shell along circumferential direction $(i=1,2,3,...,n)$

$G_{x\phi,i}$, $G_{xz,i}, G_{\phi z,i}$ = shear moduli of ith layer of multilayered shell; in-phase components of shear moduli of ith viscoelastic layer of shell

j = modal number (number of half-sine waves) along circumferential direction of multilayered shell

j_I = $\sqrt{-1}$

L = length of multilayered shell

m = modal number (number of half-sine waves) along axis of shell

n = total number of layers in multilayered shell

Q_{ij} = reduced stiffness of layers

R_i = radius of middle surface of ith layer of multilayered cylindrical shell $(i=1,2,3,...,n)$

t_i = thickness of ith layer of multilayered shell $(i=1,2,3,...,n)$

u_i, v_i = longitudinal displacements along axial and circumferential directions of different layers of shell $(i=1,2,3,...,n)$

w = radial displacement of multilayered cylindrical shell

x,ϕ,z = cylindrical shell coordinates

β_I = $(m\pi R_I)/L$

η_{ie} = material loss factor in extension for ith viscoelastic layer of multilayered shell $(i=1,2,3,...,n)$

η_{is} = material loss factor in shear for ith viscoelastic layer of multilayered shell $(i=1,2,3,...,n)$

$\nu_{x\phi,i}, \nu_{\phi x,i}$ = Poisson's ratio of ith layer of multilayered shell in x-ϕ plane $(i=1,2,3,...,n)$

ρ_i = mass density of ith layer of multilayered shell $(i=1,2,3,...,n)$

ω = resonance circular frequency of multilayered shell, rad/s

Superscripts

$(\hat{\ })$ = a complex quantity

$(\dot{\ })$ = differentiation with respect to time

$(\ )'$ = differentiation with respect to x

$(\ )^*$ = differentiation with respect to ϕ

Introduction

THE basic theory of a sandwich shell is attributed to Reissner.[1] The initial research work on the vibration problems of sandwich cylindrical shells was carried out by Yu,[2-4] who considered thickness shear deformations in the core and extensional deformations of the face layers and investigated the axisymmetric vibrations of sandwich cylindrical shells for both freely supported and infinitely long shells. Padovan and Koplic[5] derived equations for the vibration of three-layered sandwich cylindrical shells, including thickness shear deformation in both the core and face layers, and obtained solutions for free vibrations of closed and open configurations of infinite and simply supported cylindrical shells. Extensive review work on the vibration of shells has been reported by Leissa[6] and Bert and Egle.[7] The analysis of a cylindrical shell coated on one side with a damping material was first attempted by Kagawa and Krokstad.[8] Equations of motion and boundary conditions for the axisymmetric vibration of a finite length sandwich cylindrical shell with a viscoelastic core has been discussed by Pan.[9] Markus[10] investigated the axisymmetric vibration and damping of a cylindrical shell coated with viscoelastic layers on one or both sides and recently reported a refined analysis.[11]

The vibration and damping analysis of a cylindrical shell with an arbitrary number of elastic and viscoelastic layers has not been attempted so far and is the subject of the present investigation. The governing equations of motion for vibration of a general multilayered cylindrical shell having an arbitrary number of orthotropic material layers have been

Received March 4, 1982; revision received March 5, 1983.

*Reader, Department of Mechanical Engineering.
†Associate Professor, Department of Mechanical Engineering.

derived using variational principles. The strains due to extension, bending, in-plane shear, and transverse shear for all layers of the shell have been considered, and rotary and longitudinal translatory inertias along with transverse inertia have been included. A solution for a radially simply supported shell has been found (and evaluated) by taking series solutions and damping effectivenss in terms of the system loss factor for all families of the modes of vibration of a multilayered shell with elastic and viscoelastic layers.

The higher order terms considered in the present analysis give rise to higher order frequencies, which are expected to lie in a practical range for a multilayered shell with alternate elastic and viscoelastic layers; hence, their role in the damping of randomly excited shells is important. Thus, studies on the variation of higher order frequencies and the corresponding system loss factors with shell parameters are useful. Further, the present analysis may be applied to the vibration and damping analyses of layered shells of fiber-reinforced composite materials.

Equations of Motion

The cross section of an n-layered cylindrical shell is shown in Fig. 1. The curvilinear coordinate system has been employed with displacements u, v, and w along the x, ϕ, and z directions. The assumed deformation patterns in the circumferential and longitudinal directions are also shown in Fig. 1. It is assumed that the deflections are small and that the material of each layer is orthotropic. It is also assumed that the normal cross sections in each layer remain plane and continuous before and after deformation and that there is no slip at the interfaces. Such deformations account for bending, extension, and thickness shear in all of the layers and are described by the following displacement fields.

The displacements u_{zi} and v_{zi} in the ith layer in x and ϕ directions at a distance z_i from the middle of this layer is given as

$$u_{zi}=(u_i+u_{i+1})/2+z_i(u_{i+1}-u_{i-1})/t \tag{1a}$$

$$v_{zi}=(v_i+v_{i+1})/2+z_i(v_{i+1}-v_{i-1})/t \tag{1b}$$

for $i=1,2,3,\ldots,n$).

Strains in the ith layer of the shell are given as

$$\epsilon_{xx,i}=\left[u_i'\left(\frac{t_i}{2}-z_i\right)+u_{i+1}'\left(\frac{t_i}{2}+z_i\right)\right]\Big/t_i \tag{2a}$$

$$\epsilon_{\phi\phi,i}=\left[v_i^*\left(\frac{t_i}{2}-z_i\right)+v_{i+1}^*\left(\frac{t_i}{2}+z_i\right)+t_iw\right]\Big/r_it_i \tag{2b}$$

$$\gamma_{x\phi,i}=\frac{v_i'+v_{i+1}'}{2}+z_i\frac{(v_{i+1}'-v_i')}{t_i}+\frac{1}{r_i}\left[\frac{u_i^*+u_{i+1}^*}{2}+z_i\frac{(u_{i+1}^*-u_i^*)}{t_i}\right] \tag{2c}$$

$$\gamma_{xz,i}=w'+(u_{i+1}-u_i)/t_i \tag{2d}$$

$$\gamma_{\phi z,i}=\frac{w^*}{r_i}+\frac{v_{i+1}-v_i}{t_i}-\frac{1}{r_i}\left[\frac{v_{i+1}-v_{i-1}}{2}+z_i\frac{(v_{i+1}-v_{i-1})}{t_i}\right] \tag{2e}$$

where $r_i=R_i+z_i$.

Each layer of the shell is considered to be orthotropic with six independent elastic constants: for the ith layer, $E_{x,i}$, $E_{\phi,i}$, $G_{x\phi,i}$, $\nu_{x\phi,i}$, $G_{xz,i}$, and $G_{\phi z,i}$.

The stress-strain relationships for each layer can be expressed as

$$\begin{bmatrix}\sigma_{xx,i}\\ \sigma_{\phi\phi,i}\\ \tau_{x\phi,i}\end{bmatrix}=\begin{bmatrix}Q_{11,i} & Q_{12,i} & 0\\ Q_{21,i} & Q_{22,i} & 0\\ 0 & 0 & Q_{66,i}\end{bmatrix}\begin{bmatrix}\epsilon_{xx,i}\\ \epsilon_{\phi\phi,i}\\ \gamma_{x\phi,i}\end{bmatrix} \tag{3a}$$

$$\begin{bmatrix}\tau_{\phi z,i}\\ \tau_{xz,i}\end{bmatrix}=\begin{bmatrix}C_{44,i} & 0\\ 0 & C_{55,i}\end{bmatrix}\begin{bmatrix}\gamma_{\phi z,i}\\ \gamma_{xz,i}\end{bmatrix} \tag{3b}$$

where

$$Q_{11,i}=E_{x,i}/(1-\nu_{\phi x,i}\nu_{x\phi,i}),\quad Q_{22,i}=E_{\phi,i}/(1-\nu_{\phi x,i}\nu_{x\phi,i})$$

$$Q_{12,i}=\nu_{\phi x,i}E_{x,i}/(1-\nu_{\phi x,i}\nu_{x\phi,i})=Q_{21,i}=\nu_{x\phi,i}E_{\phi,i}/(1-\nu_{\phi x,i}\nu_{x\phi,i})$$

$$Q_{66,i}=G_{x\phi,i},\quad C_{44,i}=G_{\phi z,i},\quad C_{55,i}=G_{xz,i} \tag{4}$$

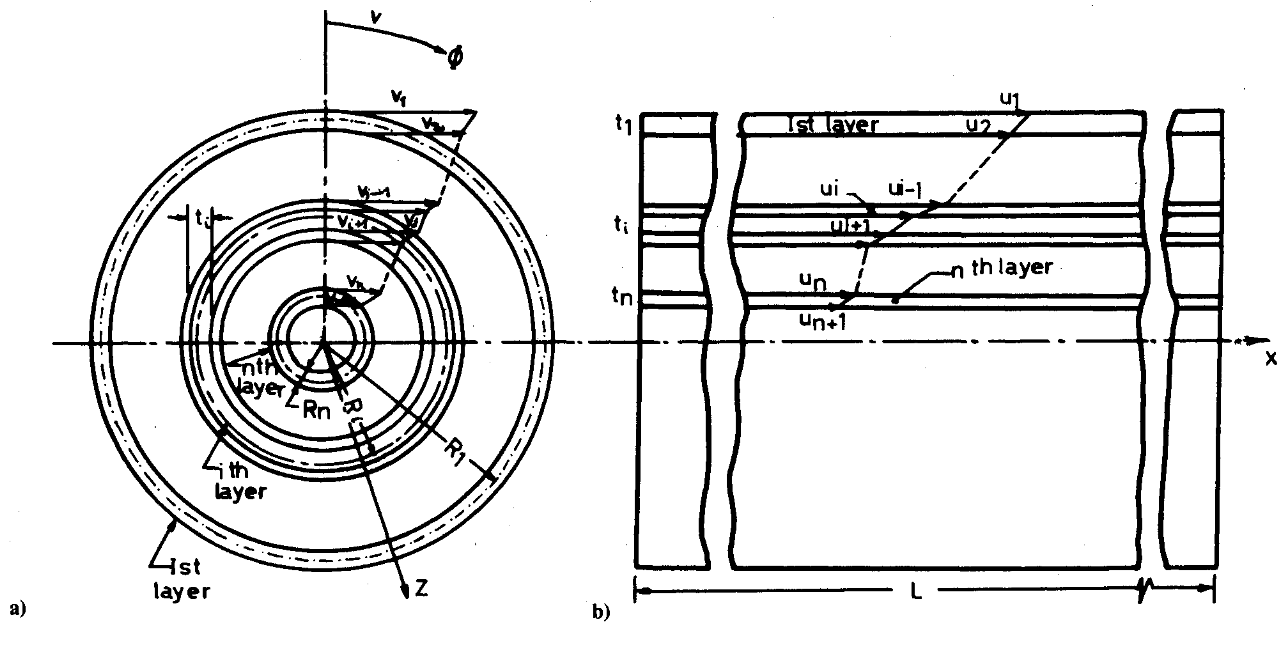

Fig. 1 a) Circumferential displacement of layers in sectional plane ϕz. b) Longitudinal displacement of layers in longitudinal plane x-z.

The strain energy U for the n-layered shell is given as

$$U=\frac{1}{2}\sum_{i=1,2}^{n}\int_0^{2\pi}\int_0^L\int_{-t_i/2}^{+t_i/2}(\sigma_{xx,i}\epsilon_{xx,i}+\sigma_{\phi\phi,i}\epsilon_{\phi\phi,i}+\tau_{x\phi,i}\gamma_{x\phi,i}+\tau_{xz,i}\gamma_{xz,i}+\tau_{\phi z,i}\gamma_{\phi z,i})\,r_i\,\mathrm{d}\phi\mathrm{d}x\mathrm{d}z_i \tag{5}$$

The total kinetic energy T of the shell is given as

$$T=\frac{1}{2}\Bigg\{\sum_{i=1,2}^{n}\int_0^{2\pi}\int_0^L\rho_i t_i R_i\dot{w}^2\mathrm{d}\phi\mathrm{d}x+\sum_{i=1,2}^{n}\int_0^{2\pi}\int_0^L\rho_i\left[R_it_i\left(\frac{\dot{u}_{i+1}+\dot{u}_i}{2}\right)^2+\frac{R_it_i}{12}(\dot{u}_{i+1}-\dot{u}_i)^2+(\dot{u}_{i+1}^2-\dot{u}_i^2)\frac{t_i^2}{12}\right]\mathrm{d}\phi\mathrm{d}x$$
$$+\sum_{i=1,2}^{n}\int_0^{2\pi}\int_0^L\rho_i\left[R_it_i\left(\frac{\dot{v}_{i+1}+\dot{v}_i}{2}\right)^2+\frac{R_it_i}{12}(\dot{v}_{i+1}-\dot{v}_i)^2+(\dot{v}_{i+1}^2-\dot{v}_i^2)\frac{t_i^2}{12}\right]\mathrm{d}\phi\mathrm{d}x\Bigg\} \tag{6}$$

The work done by the external radial excitation forces $f(x,\phi)g(t)$ is given by

$$V=\int_0^{2\pi}\int_0^L f(x,\phi)g(t)\,w\mathrm{d}\phi\mathrm{d}x \tag{7}$$

Performing the variation term by term and making use of Hamilton's principle, the following equations of motion are obtained:

$$u''_{i-1}Q_{11,i-1}\frac{R_{i-1}t_{i-1}}{6}+u''_i\left[Q_{11,i}\left(\frac{R_it_i}{3}-\frac{t_i^2}{12}\right)+Q_{11,i-1}\left(\frac{R_{i-1}t_{i-1}}{3}-\frac{t_{i-1}^2}{12}\right)\right]+u''_{i+1}Q_{11,i}\frac{R_it_i}{6}+v^{*\prime}_{i-1}\left(Q_{12,i-1}\frac{t_{i-1}}{6}+Q_{66,i-1}\frac{t_{i-1}}{6}\right)$$
$$+v_i^{*\prime}\left(Q_{12,i}\frac{t_i}{3}+Q_{12,i-1}\frac{t_{i-1}}{3}+Q_{66,i}\frac{t_i}{3}+Q_{66,i-1}\frac{t_{i-1}}{3}\right)+v^{*\prime}_{i+1}\left(Q_{12,i}\frac{t_i}{6}+Q_{66,i}\frac{t_i}{6}\right)+u^{**}_{i-1}\left[\frac{Q_{66,i-1}}{t_{i-1}^2}\left(\frac{t_{i-1}^2}{4}A_{1,i-1}-A_{3,i-1}\right)\right]$$
$$+u_i^{**}\left[\frac{Q_{66,i}}{t_i^2}\left(\frac{t_i^2}{4}A_{1,i}+A_{3,i}-t_iA_{2,i}\right)+\frac{Q_{66,i-1}}{t_{i-1}^2}\left(\frac{t_{i-1}^2}{4}A_{1,i-1}+A_{3,i-1}+t_{i-1}A_{2,i-1}\right)\right]+u^{**}_{i+1}\left[\frac{Q_{66,i}}{t_i^2}\left(\frac{t_i^2}{4}A_{1,i}-A_{3,i}\right)\right]$$
$$+w'\left(Q_{12,i}\frac{t_i}{2}+Q_{12,i-1}\frac{t_{i-1}}{2}+C_{55,i}R_i-C_{55,i-1}R_{i-1}\right)+u_{i-1}C_{55,i-1}\frac{R_{i-1}}{t_{i-1}}-u_i\left(C_{55,i}\frac{R_i}{t_i}+C_{55,i-1}\frac{R_{i-1}}{t_{i-1}}\right)+u_{i+1}C_{55,i}\frac{R_i}{t_i}$$
$$-\ddot{u}_{i-1}\frac{R_{i-1}t_{i-1}\rho_{i-1}}{6}-\ddot{u}_i\left(\frac{R_it_i\rho_i}{3}+\frac{R_{i-1}t_{i-1}\rho_{i-1}}{3}-\frac{t_i^2\rho_i}{12}+\frac{t_{i-1}^2\rho_{i-1}}{12}\right)-\ddot{u}_{i+1}\frac{R_it_i\rho_i}{6}=0 \tag{8}$$

for $i=1,2,3,\ldots,(n+1)$, these are $(n+1)$ equations.

$$v''_{i-1}Q_{66,i-1}\frac{R_{i-1}t_{i-1}}{6}+v''_i\left(Q_{66,i}\frac{R_it_i}{3}+Q_{66,i-1}\frac{R_{i-1}t_{i-1}}{3}-Q_{66,i}\frac{t_i^2}{12}+Q_{66,i-1}\frac{t_{i-1}^2}{12}\right)+v''_{i+1}Q_{66,i}\frac{R_it_i}{6}+v^{**}_{i-1}\frac{Q_{22,i-1}}{t_{i-1}^2}$$
$$\times\left(\frac{t_{i-1}^2}{4}A_{1,i-1}-A_{3,i-1}\right)+v_i^{**}\left[\frac{Q_{22,i}}{t_i^2}\left(\frac{t_i^2}{4}A_{1,i}+A_{3,i}-t_iA_{2,i}\right)+\frac{Q_{22,i-1}}{t_{i-1}^2}\left(\frac{t_{i-1}^2}{4}A_{1,i-1}+A_{3,i-1}+t_{i-1}A_{2,i-1}\right)\right]+v^{**}_{i+1}\frac{Q_{22,i}}{t_i^2}$$
$$\times\left(\frac{t_i^2}{4}A_{1,i}-A_{3,i}\right)+u^{*\prime}_{i-1}\left(Q_{12,i-1}\frac{t_{i-1}}{6}+Q_{66,i-1}\frac{t_{i-1}}{6}\right)$$
$$+u^{*\prime}_{i+1}\left(Q_{12,i}\frac{t_i}{6}+Q_{66,i}\frac{t_i}{6}\right)+u_i^{*\prime}\left(Q_{12,i}\frac{t_i}{3}+Q_{12,i-1}\frac{t_{i-1}}{3}+Q_{66,i}\frac{t_i}{3}+Q_{66,i-1}\frac{t_{i-1}}{3}\right)$$
$$+w^*\left[\frac{Q_{22,i}}{t_i}\left(\frac{t_i}{2}A_{1,i}-A_{2,i}\right)+\frac{Q_{22,i-1}}{t_{i-1}}\left(\frac{t_{i-1}}{2}A_{1,i-1}+A_{2,i-1}\right)+C_{44,i}A_{1,i}\frac{2R_i+t_i}{2t_i}-C_{44,i-1}A_{1,i-1}\frac{2R_{i-1}-t_{i-1}}{2t_{i-1}}\right]$$
$$+v_{i-1}C_{44,i-1}A_{1,i-1}\frac{4R_{i-1}^2-t_{i-1}^2}{4t_{i-1}^2}+v_i\left[C_{44,i}\frac{(2R_i+t_i)^2}{4t_i^2}A_{1,i}+C_{44,i-1}\frac{(2R_{i-1}+t_{i-1})^2}{4t_{i-1}^2}A_{1,i-1}\right]+v_{i+1}C_{44,i}\frac{(4R_i^2-t_i^2)}{4t_i^2}A_{1,i}$$
$$-\ddot{v}_{i-1}\frac{R_{i-1}t_{i-1}\rho_{i-1}}{6}-\ddot{v}_i\left(\frac{R_it_i\rho_i}{3}+\frac{R_{i-1}t_{i-1}\rho_{i-1}}{3}-\frac{t_i^2\rho_i}{12}-\frac{t_{i-1}^2\rho_{i-1}}{12}\right)-\ddot{v}_{i+1}\frac{R_it_i\rho_i}{6}=0 \tag{9}$$

for $i=1,2,3,\ldots,(n+1)$, these are $(n+1)$ equations.

$$\sum_{i=1,2}^{n}\left[(u'_i+u'_{i+1})\frac{Q_{12,i}t_i}{2}-C_{55,i}R_i(u'_{i+1}-u'_i)\right]+\sum_{i=1,2}^{n}\left\{\frac{Q_{22,i}}{t_i}v_i^*\left(\frac{t_i}{2}A_{1,i}-A_{2,i}\right)+\frac{Q_{22,i}}{t_i}v^*_{i+1}\left(\frac{t_i}{2}A_{1,i}-A_{2,i}\right)+\frac{C_{44,i}A_{1,i}}{2}\right.$$
$$\left.\times\left[v_i^*\frac{(2R_i+t_i)}{t_i}-v^*_{i+1}\frac{(2R_i-t_i)}{t_i}\right]\right\}+\sum_{i=1,2}^{n}(R_it_i\rho_i\ddot{w}-C_{55,i}t_i\rho_iw''-C_{44,i}A_{1,i}w^{**}+Q_{22,i}A_{1,i}w)+f(x,\phi)g(t)=0 \tag{10}$$

The boundary conditions obtained at $x=0$ and L are:

1) Either $w=0$ or

$$\sum_{i=1,2}^{n} C_{55,i}\left[2R_it_iw'+2R_i(u_{i+1}-u_i)\right]=0 \tag{11a}$$

2) Either $u_i=0$ or

$$Q_{11,i}t_i\left(\frac{2R_i}{3}u_i'+\frac{R_i}{3}u_{i+1}'-\frac{t_i}{6}u_i'\right)+Q_{12,i}t_i\left(\frac{v_i^*}{3}+\frac{v_{i+1}}{6}+\frac{w}{2}\right)+Q_{11,i-1}t_{i-1}\left(\frac{2R_{i-1}}{3}u_i'+\frac{R_i}{3}u_{i-1}'+\frac{u_i't_{i-1}}{6}\right)$$

$$+2Q_{12,i-1}t_{i-1}\left(\frac{v_{i-1}^*}{6}+\frac{v_i^*}{3}+\frac{w}{2}\right)+Q_{12,i}t_i\left(\frac{v_i^*}{3}+\frac{v_{i+1}^*}{6}+\frac{w}{2}\right)=0 \tag{11b}$$

for $i=1,2,3,\ldots,n+1$.

3) Either $v_i=0$ or

$$Q_{66,i}R_it_i\left(\frac{2}{3}v_i'+\frac{v_{i+1}'}{3}\right)+2Q_{66,i}t_i\left(\frac{u_i^*}{3}+\frac{u_{i+1}^*}{6}\right)-Q_{66,i}\frac{t_i^2}{6}v_i'+Q_{66,i-1}R_{i-1}t_{i-1}\left(\frac{2}{3}v_i'+\frac{v_{i-1}'}{3}\right)$$

$$+Q_{66,i}\frac{t_{i-1}^2}{6}v_i'+2Q_{66,i-1}t_{i-1}\left(u_{i-1}^*+\frac{u_i^*}{3}\right)=0 \tag{11c}$$

for $i=1,2,3,\ldots,n+1$.

For axisymmetric vibrations of a cylindrical shell, the deformations are independent of the angular coordinate. The equations of motion for axisymmetric vibration of a multilayered shell may be written from Eqs. (8-10) as follows:

$$u_{i-1}''Q_{11,i-1}\frac{t_{i-1}R_{i-1}}{6}+u_i''\left[Q_{11,i}\left(\frac{R_it_i}{3}-\frac{t_i^2}{3}\right)+Q_{11,i-1}\left(\frac{R_{i-1}t_{i-1}}{3}+\frac{t_{i-1}^2}{12}\right)\right]+u_{i+1}''Q_{11,i}\frac{R_it_i}{6}+u_{i-1}C_{55,i-1}\frac{R_{i-1}}{t_{i-1}}$$

$$-u_i\left(C_{55,i}\frac{R_i}{t_i}+C_{55,i-1}\frac{R_{i-1}}{t_{i-1}}\right)+u_{i+1}C_{55,i}\frac{R_i}{t_i}+w'\left(Q_{12,i}\frac{t_i}{2}+Q_{12,i-1}\frac{t_{i-1}}{2}+C_{55,i}R_i-C_{55,i-1}R_{i-1}\right)-\ddot{u}_{i-1}\frac{R_{i-1}t_{i-1}\rho_{i-1}}{6}$$

$$-\ddot{u}_i\left(\frac{R_it_i\rho_i}{3}+\frac{R_{i-1}t_{i-1}\rho_{i-1}}{3}-\frac{\rho_it_i^2}{12}+\frac{\rho_{i-1}t_{i-1}^2}{12}\right)-\ddot{u}_{i+1}\frac{R_it_i\rho_i}{6}=0 \tag{12}$$

For $i=1,2,3,\ldots,(n+1)$, this gives $(n+1)$ equations,

$$v_{i-1}''Q_{66,i-1}\frac{R_{i-1}t_{i-1}}{6}+v_i''\left(Q_{66,i}\frac{R_it_i}{3}-Q_{66,i}\frac{t_i^2}{12}+Q_{66,i-1}\frac{R_{i-1}t_{i-1}}{3}+Q_{66,i-1}\frac{t_{i-1}^2}{12}\right)+v_{i+1}''Q_{66,i}\frac{R_it_i}{6}$$

$$+v_{i-1}C_{44,i-1}A_{1,i-1}\frac{(4R_{i-1}^2-t_{i-1}^2)}{4t_{i-1}^2}-v_i\left[C_{44,i}A_{1,i}\frac{(2R_i+t_i)^2}{4t_i^2}+C_{44,i-1}A_{1,i-1}\frac{(2R_{i-1}-t_{i-1})^2}{4t_{i-1}^2}\right]+v_{i+1}\left[C_{44,i}A_{1,i}\frac{(4R_i^2-t_i^2)}{4t_i^2}\right]$$

$$-\ddot{v}_{i-1}\frac{R_{i-1}t_{i-1}\rho_{i-1}}{6}-\ddot{v}_i\left(\frac{R_it_i\rho_i}{3}+\frac{R_{i-1}t_{i-1}\rho_{i-1}}{3}-\frac{\rho_{i-1}t_{i-1}^2}{12}-\frac{\rho_it_i^2}{12}\right)-\ddot{v}_{i+1}\frac{R_it_i\rho_i}{6}=0 \tag{13}$$

For $i=1,2,3,\ldots,n+1$, this gives $(n+1)$ equations and

$$\sum_{i=1,2}^{n}\left[(u_i'+u_{i+1}')\frac{Q_{12,i}t_i}{2}-C_{55,i}R_i(u_{i+1}'-u_i')\right]+\sum_{i=1,2}^{n}(R_it_i\rho_i\ddot{w}-C_{55,i}R_it_iw''+Q_{22,i}A_{1,i}w)+f(x)g(t)=0 \tag{14}$$

Equations (12) and (14) are coupled and represent equations of motion for coupled radial and longitudinal motion. Equation (13) is uncoupled and represents torsional motion.

In the above equations

$$A_{1,i}=\int_{-t_i/2}^{+t_i/2}\frac{\mathrm{d}z_i}{(R_i+z_i)}=\log_e\left[\frac{R_i+(t_i/2)}{R_i-(t_i/2)}\right]$$

$$A_{2,i}=\int_{-t_i/2}^{+t_i/2}\frac{z_i\mathrm{d}z_i}{(R_i+z_i)}=t_i-R_i\log_e\left[\frac{R_i+(t_i/2)}{R_i-(t_i/2)}\right]$$

$$A_{3,i}=\int_{-t_i/2}^{+t_i/2}\frac{z_i^2\mathrm{d}z_i}{(R_i+z_i)}=-R_it_i+R_i^2\log_e\left[\frac{R_i+(t_i/2)}{R_i-(t_i/2)}\right]$$

For $t_i/R_i \ll 1$, after expansion and neglecting the higher-order terms,

$$A_{1,i}=\frac{t_i}{R_i}+\frac{1}{12}\left(\frac{t_i}{R_i}\right)^3 \tag{15a}$$

$$A_{2,i}=-\frac{1}{12}\frac{t_i^3}{R_i^2}-\frac{1}{80}\frac{t_i^5}{R_i^4} \tag{15b}$$

$$A_{3,i}=\frac{1}{12}\frac{t_i^3}{R_i}+\frac{1}{80}\frac{t_i^5}{R_i^3} \tag{15c}$$

Solution for Finite Cylindrical Shell with Simply Supported Ends

The solution for a radially simply supported multilayered cylindrical shell consisting of elastic stiff and elastic soft layers was determined first. Subsequently, the vibration and damping analysis of a multilayered shell with constrained viscoelastic layers was obtained from the solution for all of elastic layers by replacing alternate soft elastic layers with viscoelastic layers. The properties of the viscoelastic layers have been taken by the principle of correspondence of linear viscoelasticity, i.e., under harmonic motion, the moduli of viscoelastic materials have been taken as complex quantities.

The series solution of the vibrational response at resonant frequencies satisfies the differential equations (8-10) for the nonaxisymmetric vibration of a multilayered cylindrical shell as well as the boundary conditions in restricted manner as follows:

$$u_i = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} U_{mj,i} \cos\frac{m\pi x}{L} \sin j\phi \sin\omega t \tag{16a}$$

$$v_i = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} V_{mj,i} \sin\frac{m\pi x}{L} \sin j\phi \sin\omega t \tag{16b}$$

$$w = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} W_{mj,i} \sin\frac{m\pi x}{L} \sin j\phi \sin\omega t \tag{16c}$$

Expanding the excitation force results in

$$f(x,\phi)g(t) = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} f_{mj} \sin\frac{m\pi x}{L} \sin j\phi \sin\omega t \tag{17}$$

Similarly, a series solution in the form given below satisfies the differential equations (12-14) for axisymmetric vibration of shell:

$$u_i = \sum_{m=1}^{\infty} U_{m0,i} \cos\frac{m\pi x}{L} \sin\omega t \tag{18a}$$

$$v_i = \sum_{m=1}^{\infty} V_{m0,i} \sin\frac{m\pi x}{L} \sin\omega t \tag{18b}$$

$$w = \sum_{m=1}^{\infty} W_{m0,i} \sin\frac{m\pi x}{L} \sin\omega t \tag{18c}$$

The excitation force in this case may be expanded as

$$f(x)g(t) = \sum_{m=1}^{\infty} f_m \sin\frac{m\pi x}{L} \sin\omega t \tag{19}$$

Substituting Eqs. (16) and (17) into the governing differential equations for the nonaxisymmetric vibration of a multilayered shell, $(2n+3)$ simultaneous algebraic equations are obtained. Similarly, substitution of Eqs. (18) and (19) into the equations of motion of the axisymmetric vibration of shell also gives $(2n+3)$ simultaneous algebraic equations.

The vibration and damping analysis of a multilayered cylindrical shell is obtained from the solution of all elastic layers by replacing the elastic layers with viscoelastic layers. Let η_{is} be the material loss factor of the ith layer in the in-plane and transverse shears and η_{ie} be the material loss factor in the extension along the x and ϕ directions. Thus

$$\hat{G}_{x\phi,i} = G_{x\phi,i}(1+j_1\eta_{is}), \quad \hat{G}_{xz,i} = G_{xz,i}(1+j_1\eta_{is}),$$
$$\hat{G}_{\phi z,i} = G_{\phi z,i}(1+j_1\eta_{is}), \quad \hat{E}_{xi} = E_{xi}(1+j_1\eta_{ie}),$$
$$\hat{E}_{yi} = E_{yi}(1+j_1\eta_{ie}) \tag{20}$$

for $i = 1,2,3,\ldots,n$ and $j_1 = \sqrt{-1}$.

After substituting the above complex elastic moduli in the algebraic equations obtained for nonaxisymmetric vibration and eliminating $U_{mj,i}$ and $V_{mj,i}$ $(i = 1,2,3,\ldots,n+1)$ in terms of W_{mj}, the resulting equation in W_{mj} can be written as

$$W_{mj}[(R_1+j_1I_1-\lambda_1)(R_2+j_1I_2-\lambda_2),\ldots,(2n+3)\text{terms}] = f'_{mj} \tag{21}$$

where $f'_{mj} = f_{mj}/E_{x,1}$.

Here the complex frequency parameters $\lambda_1, \lambda_2, \ldots, \lambda_{2n+3}$ define the resonance frequencies and the associated loss factors of the $(2n+3)$ modes of the family of modes corresponding to a set of values of j and $(m\pi R_1)/L$. These complex frequency parameters are also the eigenvalues of the complex matrix

$$[\hat{A}_1 + \lambda B_1]X = 0 \tag{22}$$

where

$$X = [U_{mj,1}, U_{mj,2}, \ldots, U_{mj,n+1}, V_{mj,1}, V_{mj,2}, \ldots, V_{mj,n+1}, W_{mj}]^T. \tag{23}$$

Both $[\hat{A}_1]$ and $[B_1]$ matrices are of order $(2n+3)$ and are given in the Appendix; here, $\lambda = (\rho_1 t_1 R_1 \omega^2)/E_{x,1}$.

Similarly, substitution of the complex elastic moduli of the viscoelastic layers in Eqs. (12) and (14) forms complex eigenvalue problems of the type

$$[\hat{A}_2 - \lambda B_2]X_1 = 0 \tag{24}$$

where

$$X_1 = [U_{m0,1}, U_{m0,2}, \ldots, U_{m0,n+1}, W_{m0}]^T \tag{25}$$

$[\hat{A}_2]$ and $[B_2]$ are matrices of order $(n+2)$. The eigenvalues of Eq. (24) give the resonance frequencies and associated system loss factors for the coupled radial and longitudinal modes of vibration for axisymmetric vibration. Equations (13) form a system of uncoupled equations and thus a separate eigenvalue problem of order $(n+1)$ as given below and also give resonance frequencies and associated system loss factors for torsional and other circumferential shear modes for axisymmetric vibration,

$$[\hat{A}_3 - \lambda B_3]X_2 = 0 \tag{26}$$

where

$$X_2 = [V_{m0,1}, V_{m0,2}, \ldots, V_{m0,n+1}, W_{m0}]^T \tag{27}$$

These matrices are given in the Appendix.

Conclusion

The governing differential equations of motion for nonaxisymmetric and axisymmetric vibrations of a general multilayered cylindrical shell with an arbitrary number of orthotropic material layers have been derived using variational principles. Deformations due to extension, bending, in-plane shear, and transverse shear are considered in all layers of the shell. Rotary and longitudinal translatory inertias along with transverse inertia are included. Apart from the determination of higher-order frequencies, the analysis is expected to give accurate results for the lower modes of vibration in cylindrical shells. By applying this analysis and using the corresponding principle of linear viscoelasticity, the resonant frequencies and associated system loss factors have been evaluated for radially simply supported multilayered shells consisting of alternate elastic and viscoelastic layers. For an n-layered shell, $(2n+3)$ resonant frequencies and associated loss factors have been determined for nonaxisymmetric as well as axisymmetric vibrations. For nonaxisymmetric vibrations all of the $(2n+3)$ modes are coupled, while for axisymmetric vibrations the radial mode is coupled with $(n+1)$ longitudinal modes and the remaining $(n+1)$ modes are torsional and circumferential shear modes. In the present refined analysis, each layer is of orthotropic material and its transverse shear deformation has been considered along with other deformations. As such, the analysis has a great scope of application for vibration and damping analysis of fiber-reinforced composite material shells.

Computation and numerical results of the resonating frequencies and associated system loss factors for multilayered shells are reported in Part II of the paper.

Appendix

The elements of matrices $\hat{A}_1$, B_1, $\hat{A}_2$, B_2, $\hat{A}_3$, and B_3 (see Fig. A1) are as follows:

$$GA_{(i)} = \frac{q_{11,i-1}T_{1,i-1}}{6} + q_{66,i-1}T_{2,i-1}j^2 - C_{55,i-1}T_{3,i-1}$$

$$HA_{(i)} = \frac{T_{4,i-1}}{6}$$

$$GB_{(i)} = q_{11,i}\left(\frac{T_{1,i}}{3} - \frac{T_{8,i}^2}{12}\right) + q_{11,i-1}\left(\frac{T_{1,i-1}}{3} + \frac{T_{8,i-1}^2}{12}\right)$$

$$+ q_{66,i}j^2T_{5,i} + q_{66,i-1}j^2T_{6,i-1} + C_{55,i}T_{3,i} + C_{55,i-1}T_{3,i-1}$$

$$HB_{(i)} = -\frac{T_{4,i}}{3} - \frac{T_{4,i-1}}{3} - \frac{T_{7,i-1}}{12} + \frac{T_{7,i}}{12}$$

$$GC_{(i)} = \frac{q_{11,i}T_{1,i}}{6} + q_{66,i}j^2T_{2,i} - C_{55,i}T_{3,i}$$

$$HC_{(i)} = -\frac{T_{4,i}}{6}$$

$$GD_{(i)} = \frac{q_{12,i}T_{8,i}j}{6} + \frac{q_{66,i-1}T_{8,i-1}j}{6}$$

$$GE_{(i)} = \frac{q_{12,i}T_{8,i}j}{3} + \frac{q_{12,i-1}T_{8,i-1}j}{3} + \frac{q_{66,i}T_{8,i}j}{3} + \frac{q_{66,i-1}T_{8,i-1}j}{3}$$

$$GF_{(i)} = \frac{q_{12,i}T_{8,i}j}{6} + \frac{q_{66,i}T_{8,i}j}{6}$$

$$GG_{(i)} = -\frac{q_{12,i}T_{8,i}}{2} - \frac{q_{12,i-1}T_{8,i-1}}{2}$$

$$GH_{(i)} = \frac{q_{12,i-1}T_{8,i-1}j}{6} + \frac{q_{66,i-1}T_{8,i-1}j}{6}$$

$$GI_{(i)} = \frac{q_{12,i}T_{8,i}j}{3} + \frac{q_{12,i-1}T_{8,i-1}j}{3}$$

$$GJ_{(i)} = \frac{q_{12,i}T_{8,i}j}{6} + \frac{q_{66,i}T_{8,i}j}{6}$$

$$GK_{(i)} = \frac{q_{66,i-1}T_{1,i-1}}{6} + q_{22,i-1}T_{2,i-1}j^2 - C_{44,i-1}a_{1,i-1}$$

$$\times (T_{3,i-1}^2 - 1/4)$$

$$HK_{(i)} = -\frac{T_{4,i-1}}{6}$$

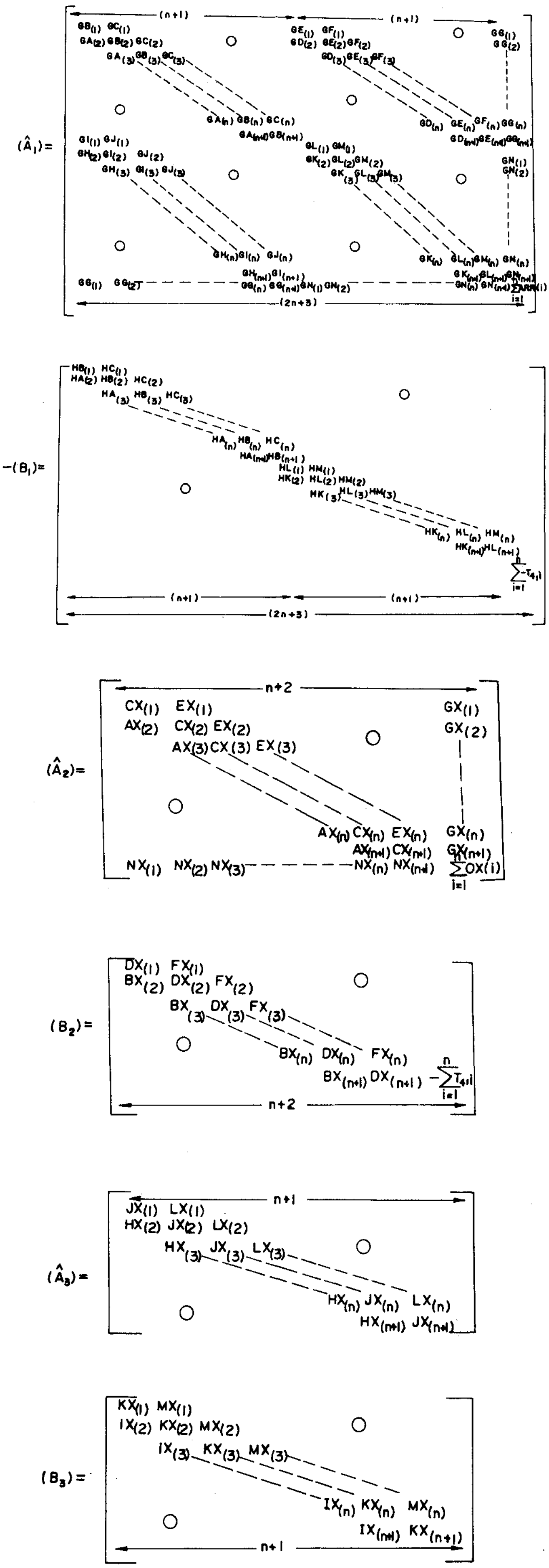

Fig. A1 Matrices $\hat{A}_1$, B_1, $\hat{A}_2$, B_2, $\hat{A}_3$, and B_3.

$$GL_{(i)} = -\frac{q_{66,i}T_{8,i}^2}{12} + \frac{q_{66,i}T_{1,i}}{3} + \frac{q_{66,i-1}T_{8,i-1}^2}{12} + \frac{q_{66,i-1}T_{1,i-1}}{3}$$
$$+ (q_{22,i}T_{5,i} + q_{22,i-1}T_{6,i-1})j^2 + C_{44,i}a_{1,i}(T_{3,i} + 1/2)^2$$
$$+ C_{44,i-1}a_{1,i-1}(T_{3,i-1} - 1/2)^2$$

$$HL_{(i)} = -\frac{T_{4,i}}{3} - \frac{T_{4,i-1}}{3} - \frac{T_{7,i-1}}{12} + \frac{T_{7,i}}{12}$$

$$GM_{(i)} = \frac{q_{66,i}T_{1,i}}{6} + q_{22,i}T_{2,i}j^2 - C_{44,i}a_{1,i}(T_{3,i}^2 - 1/4)$$

$$HM_{(i)} = -\frac{T_{4,i}}{6}$$

$$GN_{(i)} = -q_{22,i}T_{9,i}j - q_{22,i-1}T_{10,i-1}j$$
$$- C_{44,i}a_{1,i}j(T_{3,i} + 1/2) + C_{44,i-1}a_{1,i-1}j(T_{3,i-1} - 1/2)$$

$$ARR_{(i)} = C_{55,i}T_{1,i} + (C_{44,i}a_{1,i})^2 + q_{22,i}a_{1,i}$$

$$AX_{(i)} = \frac{q_{11,i-1}T_{1,i-1}}{6} - C_{55,i-1}T_{3,i-1}$$

$$BX_{(i)} = -\frac{T_{4,i-1}}{6}$$

$$CX_{(i)} = q_{11,i}\left(\frac{T_{1,i}}{3} - \frac{T_{8,i}^2}{12}\right) + q_{11,i-1}\left(\frac{T_{1,i-1}}{3} + \frac{T_{8,i-1}^2}{12}\right)$$
$$+ C_{55,i}T_{3,i} + C_{55,i-1}T_{3,i-1}$$

$$DX_{(i)} = -\frac{T_{4,i}}{3} - \frac{T_{4,i-1}}{3} - \frac{T_{7,i-1}}{12} + \frac{T_{7,i}}{12}$$

$$EX_{(i)} = \frac{q_{11,i}T_{1,i}}{6} - C_{55,i}T_{3,i}$$

$$FX_{(i)} = -\frac{T_{4,i}}{6}$$

$$GX_{(i)} = -\frac{q_{12,i}T_{8,i}}{2} - \frac{q_{12,i-1}T_{8,i-1}}{2} - C_{55,i}\bar{R}_i\beta_1$$
$$+ C_{55,i-1}\bar{R}_{i-1}\beta_1$$

$$HX_{(i)} = \frac{q_{66,i-1}T_{1,i-1}}{6} - C_{44,i-1}a_{1,i-1}(T_{3,i-1}^2 - 1/4)$$

$$IX_{(i)} = -\frac{T_{4,i-1}}{6}$$

$$JX_{(i)} = \frac{q_{66,i}T_{1,i}}{3} - \frac{q_{66,i}(T_{8,i})^2}{12} + \frac{q_{66,i-1}T_{8,i-1}^2}{12}$$
$$+ \frac{q_{66,i-1}T_{1,i-1}}{3} + C_{44,i-1}a_{1,i}(T_{3,i} + 1/2)^2$$
$$+ C_{44,i-1}a_{1,i-1}(T_{3,i-1} - 1/2)^2$$

$$KX_{(i)} = -\frac{T_{4,i}}{3} - \frac{T_{4,i-1}}{3} - \frac{T_{7,i-1}}{12} + \frac{T_{7,i}}{12}$$

$$LX_{(i)} = \frac{q_{66,i}T_{1,i}}{6} - C_{44,i}a_{1,i}(T_{3,i}^2 - 1/4)$$

$$MX_{(i)} = -\frac{T_{4,i}}{6}$$

$$NX_{(i)} = -\frac{q_{12,i}T_{8,i}}{2} - \frac{q_{12,i-1}T_{8,i-1}}{2} - C_{55,i}\bar{R}_i\beta_1$$
$$+ C_{55,i-1}\bar{R}_{i-1}\beta_1$$

$$OX_{(i)} = C_{55,i}T_{1,i} + q_{22,i}a_{1,i}$$

(for $i = 1,2,3,\ldots, n+1$)

where

$$T_{1,i} = R_i\theta_i\beta_1\beta_2, \qquad T_{2,i} = \frac{a_{1,i}}{4} - a_{3,i}$$

$$T_{3,i} = \frac{\bar{R}_i\beta_1}{\theta_i\beta_2}, \qquad T_{4,i} = \gamma_i\theta_i\bar{R}_i$$

$$T_{5,i} = \frac{a_{1,i}}{4} + a_{3,i} - a_{2,i}, \qquad T_{6,i} = \frac{a_{1,i}}{4} + a_{3,i} + a_{2,i}$$

$$T_{7,i} = \frac{\gamma_i\theta_i^2\beta_2}{\beta_1}, \qquad T_{8,i} = \theta_i\beta_2$$

$$T_{9,i} = \frac{a_{1,i}}{2} - a_{2,i}, \qquad T_{10,i} = \frac{a_{1,i}}{2} + a_{2,i}$$

(for $i = 1,2,3,\ldots,n$)

and

$$a_{1,i} = A_{1,i} = \frac{\theta_i\gamma_s}{\bar{R}_i} + \left(\frac{\theta_i\gamma_s}{\bar{R}_i}\right)^3\frac{1}{12}$$

$$a_{2,i} = \frac{A_{2,i}}{t_i} = -\left(\frac{\theta_i\gamma_s}{\bar{R}_i}\right)^2\frac{1}{12} - \left(\frac{\theta_i\gamma_s}{\bar{R}_i}\right)^4\frac{1}{80}$$

$$a_{3,i} = \frac{A_{3,i}}{t_i^2} = \frac{\theta_i\gamma_s}{12\bar{R}_i} + \left(\frac{\theta_i\gamma_s}{\bar{R}_i}\right)^3\frac{1}{80}$$

$$q_{11,i} = \frac{Q_{11,i}}{E_{x,1}} = \frac{\alpha_i}{(1-\psi_i^2\nu_{\phi x,1}/\tau_i)}$$

$$q_{22,i} = \frac{Q_{22,i}}{E_{x,1}} = \frac{\alpha_i}{(1-\psi_i^2\nu_{\phi x,1}/\tau_i)\phi_i}$$

$$q_{12,i} = \frac{Q_{12,i}}{E_{x,1}} = \frac{\psi_i\nu_{\phi x,i}\alpha_i}{(1-\psi_i^2\nu_{\phi x,1}/\tau_i)}$$

$$q_{66,i} = \frac{Q_{66,i}}{E_{x,1}} = \delta_i'$$

$$C_{44,i} = \frac{C_{44,i}}{E_{x,1}} = \frac{\delta_i}{e_i}$$

$$C_{55,i} = \frac{C_{55,i}}{E_{x,i}} = \delta_i$$

where

$$\bar{R}_i = \frac{R_i}{R_1}, \qquad \gamma_s = \frac{t_1}{R_1}, \qquad e_i = \frac{G_{xz,i}}{G_{x,\phi,i}}, \quad \beta_2 = \gamma_s\beta_1$$

$$\alpha_i = \frac{E_{x,i}}{E_{x,1}}, \qquad \gamma_i = \frac{\rho_i}{\rho_1}, \qquad \delta_i = \frac{G_{xz,i}}{E_{x,1}}$$

$$\delta_i' = \frac{G_{x,\phi,i}}{E_{x,1}}, \qquad \theta_i = \frac{t_i}{t_1}, \qquad \phi_i = \frac{E_{x,i}}{E_{\phi,i}}$$

$$\psi_i = \frac{\nu_{\phi x,i}}{\nu_{\phi x,1}}, \qquad \tau_i = \frac{\nu_{\phi x,i}}{\nu_{x\phi,i}}$$

(for $i = 1,2,3,\ldots,n$)

References

[1]Reissner, E., "Small Bending and Stretching of Sandwich Type Shells," NACA Rept. 975, 1960 (supersedes NACA TN 1832, 1949).

[2]Yu, Y. Y., "Vibration of Elastic Sandwich Cylindrical Shells," *Journal of Applied Mechanics*, Vol. 27, Dec. 1960, p. 653.

[3]Yu, Y. Y., "Application of Variational Equation of Motion to the Non-Linear Analysis of Homogeneous and Layered Plates and Shells," AFOSR TN 2256, 1962.

[4]Yu, Y. Y., "Viscoelastic Damping of Vibration of Sandwich Plates and Shells," Paper presented at IASS Symposium on Non-Classical Shell Problems, 1963.

[5]Padovan, J. and Koplik, B., "Vibrations of Closed and Open Sandwich Cylindrical Shells Using Refined Theory," *Journal of Acoustical Society of America*, Vol. 47, No. 3 (Pt. 2), 1970, pp. 862-869.

[6]Leissa, A. W., "Vibration of Shells," NASA SP-288, 1973.

[7]Bert, C. W., Baker, J. L., and Egle, D. M., "Free Vibration of Multilayer Anisotropic Cylindrical Shells," *Journal of Composite Materials*, Vol. 3, July 1969, pp. 480-499; Errata, Vol. 4, April 1970, p. 283.

[8]Kagawa, Y. and Krokstad, A., "On Damping of Cylindrical Shells Coated with Viscoelastic Materials," ASME Pub. 69-Vibr.-9, 1960, pp. 1-9.

[9]Pan, H. H., "Axisymmetric Vibration of a Circular Sandwich Shell with Viscoelastic Core Layer," *Journal of Sound and Vibration*, Vol. 9(2), 1969, pp. 338-348.

[10]Markus, S., "Damping Properties of Layered Cylindrical Shells Vibrating in Axially Symmetric Modes," *Journal of Sound and Vibration*, Vol. 48, No. 4, 1976, pp. 511-524.

[11]Markus, S., "Refined Theory of Damped Axisymmetric Vibrations of Double-Layered Cylindrical Shells," *Journal of Mechanical Engineering Science*, Vol. 21, No. 1, 1970, pp. 33-37.